Received: *February 2, 2001*, Accepted: *March 29, 2001*
Revised: *March 27, 2001*

Two-black-hole bound states

Ruth Britto-Pacumio, Andrew Strominger and Anastasia Volovich*

Department of Physics, Harvard University
Cambridge, MA 02138
E-mail: britto@boltzmann.harvard.edu, andy@planck.harvard.edu, nastya@physics.harvard.edu

Abstract: The quantum mechanics of N slowly-moving BPS black holes in five dimensions is considered. A divergent continuum of states describing arbitrarily closely bound black holes with arbitrarily small excitation energies is found. A superconformal structure appears at low energies and can be used to define an index counting the weighted number of supersymmetric bound states. It is shown that the index is determined from the dimensions of certain cohomology classes on the symmetric product of N copies of $\mathbb{R}^4$. An explicit computation for the case of $N = 2$ with no angular momentum yields a finite nonzero result.

Keywords: Black Holes in String Theory, Black Holes, Supersymmetric Effective Theories.

*On leave from L.D. Landau Institute for Theoretical Physics Moscow, Russia

Contents

JHEP03(2001)050

1. Introduction

The low energy dynamics of N near-coincident BPS black holes in five dimensions is described by superconformal quantum mechanics [1]–[4]. (See also [5]–[10].) The states of this theory describe black holes whose separations and excitation energies go to zero in the infrared scaling limit. Hence every state of this low energy theory describes a marginally bound state of the N black holes.

The existence of infinitely many bound states is puzzling. In this paper we compute a supersymmetric index which counts a weighted number of bound states (roughly the difference between the numbers of hypermultiplets and vector multiplets). This index, in contrast, turns out to be finite and nonvanishing. After factoring out center-of-mass degrees of freedom, the index essentially reduces to the Witten index of the black hole quantum mechanics:

$$\mathcal{I}_{BH}(j_L) \equiv \mathrm{Tr}(-)^{2J_R^3}|_{j_L}. \tag{1.1}$$

In this expression J_R is the generator of $\mathrm{SU}(2)_R$ spatial rotations and is a generator of the superconformal algebra. $\mathrm{SU}(2)_L$ spatial rotations commute with the superconformal algebra, and the trace in (1.1) is at fixed total j_L. Ordinarily this would be computed as a trace over eigenstates of the hamiltonian H. This trace is ill-defined for superconformal quantum mechanics because of the infrared continuum. (The usual method of putting the system in a box does not work here because the continuum arises from near coincident black holes.) We define the index by tracing instead over eigenstates of $L_0 = \frac{1}{2}(H + K)$, where K is the generator of special conformal transformations. For the case of two black holes we find that $\mathcal{I}_{BH}(0) = -1$. For general N we relate the index to the counting of a certain type of noncompact cohomology class in the symmetric (for identical black holes) product of N copies of $\mathbb{R}^4$. At higher N there appears to be a rich structure of supersymmetric bound states whose elucidation we defer to later work.

On the way to studying the N-black hole problem we describe some general properties of the Hilbert space of superconformal quantum mechanics. We begin with the simplest case with two supersymmetries and work up to black holes. In section 2 we relate the spectrum of an $\mathrm{OSp}(1|2)$ sigma model to the eigenvalues of a certain Dirac operator on the target space. In section 3 we show that states of $\mathrm{SU}(1,1|1)$ superconformal quantum mechanics are naturally viewed as $(p,0)$-forms, but with a non-canonical measure, and we derive the chiral primary condition. In section 4 we describe the $D(2,1;0)$ quantum mechanics. In section 5 we explicitly compute the index $\mathcal{I}_{BH}(0)$ for two black holes.

2. OSp(1|2)

In this section we relate the spectrum of an $\mathrm{OSp}(1|2)$ sigma model to the eigenvalues of a certain Dirac operator with torsion on the target space.

2.1 Symmetries

The $\mathrm{OSp}(1|2)$ superalgebra is generated by the three bosonic operators H, K and D and two fermionic operators Q and S. The nonvanishing commutation relations are

$$\begin{aligned}
&[H,K] = -iD\,, \qquad &&[H,D] = -2iH\,, \qquad &&[K,D] = 2iK\,,\\
&\{Q,Q\} = 2H\,, &&[Q,D] = -iQ\,, &&[Q,K] = -iS\,,\\
&\{S,S\} = 2K\,, &&[S,D] = iS\,, &&[S,H] = iQ\,,\\
&\{S,Q\} = D\,.
\end{aligned} \tag{2.1}$$

Alternatively one may define (suppressing an arbitrary dimensionful constant)

$$
\begin{aligned}
G_{\pm 1/2} &= \frac{1}{\sqrt{2}}(Q \mp iS)\,,\\
L_0 &= \frac{1}{2}(H+K)\,,\\
L_{\pm 1} &= \frac{1}{2}(H-K\mp iD)\,,
\end{aligned}
\tag{2.2}
$$

in terms of which the algebra becomes

$$
\begin{aligned}
[L_m, G_r] &= \frac{m-2r}{2}G_{m+r}\,,\\
\{G_r, G_s\} &= 2L_{r+s}\,,\\
[L_m, L_n] &= (m-n)L_{m+n},
\end{aligned}
\tag{2.3}
$$

where $r,s=\pm 1/2$ and $m,n=0,\pm 1$.

The OSp(1|2) algebra can be realized with the supermultiplet (X^M,λ^M), where $\lambda^M=\lambda^{M\dagger}$. The nonvanishing commutation relations for these fields and their conjugate momenta are, in the notation of [11],

$$
\begin{aligned}
\{\lambda^M, \lambda^N\} &= g^{MN}\,,\\
[P_M, X^N] &= -i\delta_M^{\ N}\,,\\
[P_M, \lambda^N] &= i\left(\Gamma^N_{\ MP} - \omega_M{}^N{}_P\right)\lambda^P\,,
\end{aligned}
\tag{2.4}
$$

with ω the spin connection and Γ the Christoffel connection. The last relation is necessitated by the fact that the commutator of λ with itself depends on X, and implies that $e^\alpha{}_M\lambda^M$, where e is the vielbein, commutes with P. Explicit expressions for the supercharges are then

$$
Q = \lambda^M P_M + \frac{i}{6}(c_{MNP} - 3\omega_{MNP})\lambda^M\lambda^N\lambda^P\,,
\tag{2.5}
$$

and

$$
S = \lambda^M D_M\,.
\tag{2.6}
$$

The algebra requires that the vector field D is a so-called closed homothety obeying

$$
\begin{aligned}
\mathcal{L}_D g_{MN} &= 2g_{MN}\,,\\
d(D_M dX^M) &= 0\,,
\end{aligned}
\tag{2.7}
$$

and the torsion c obeys

$$
\begin{aligned}
D^M c_{MNP} &= 0\,,\\
\mathcal{L}_D c_{MNP} &= 2c_{MNP}\,.
\end{aligned}
\tag{2.8}
$$

A derivation of these results can be found in [11].

2.2 Quantum states

States $|\psi\rangle$ in the Hilbert space form a representation of the algebra of fermions. From the commutation relations (2.4), we conclude that the states are target space spinors, with the fermions acting as

$$\lambda^M|\psi\rangle = \frac{1}{\sqrt{2}}\gamma^M|\psi\rangle\,, \tag{2.9}$$

where γ^M are the usual SO(N) gamma matrices.

The states can be organized into infinite-dimensional superconformal multiplets. At the bottom of every multiplet is a superprimary state obeying

$$G_{1/2}|\psi\rangle = 0\,. \tag{2.10}$$

The remaining tower of states is generated by the action of $G_{-1/2}$. Superprimary states are related by a similarity transformation to states $|\psi'\rangle = e^K|\psi\rangle$ obeying

$$Q|\psi'\rangle = 0\,. \tag{2.11}$$

Using (2.9) and (2.11) becomes the modified Dirac equation

$$-i\left(\gamma^M\nabla_M - \frac{1}{12}\gamma_M{}^{NP}c^M{}_{NP}\right)|\psi'\rangle = 0\,. \tag{2.12}$$

In order to better understand (2.12), we introduce the conformally related metric

$$d\tilde{s}^2 = K^{-1}ds^2\,. \tag{2.13}$$

It follows from (2.7) that the vector field D is covariantly constant in this metric. Hence coordinates can be chosen so that it takes the simple product form

$$d\tilde{s}^2 = 2(dX^0)^2 + \hat{g}_{IJ}dX^I dX^J\,, \tag{2.14}$$

where the $(N-1)$-dimensional metric $\hat{g}$ on the space M^{N-1} transverse to the orbits of D is independent of X^0. In these coordinates, $D^M\partial_M = \partial_0$ and $K = \frac{1}{2}e^{2X^0}$. Equation (2.11) then reduces to

$$-i\left(\gamma^I\hat{\nabla}_I - \frac{1}{12}\gamma_I{}^{JK}c^I{}_{JK}\right)|\psi'\rangle = i\gamma^0\left(\partial_0 + \frac{N-1}{2}\right)|\psi'\rangle\,. \tag{2.15}$$

The left hand side of (2.15) is a Dirac equation with torsion on the transverse space M^{N-1}. Let λ_j denote the (real) spectrum of this Dirac operator and $\hat{\psi}_j$ the corresponding orthonormal eigenspinors. In an even-dimensional space, there is a basis in which

$$\gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \gamma^I & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}, \tag{2.16}$$

and a solution of the Dirac equation can be written as

$$|\psi_i'\rangle = \begin{pmatrix} \psi_U' \\ \psi_L' \end{pmatrix} = \begin{pmatrix} \hat{\psi}_i e^{-(\frac{N-1}{2}+\lambda_i)X^0} \\ \hat{\psi}_i e^{-(\frac{N-1}{2}-\lambda_i)X^0} \end{pmatrix}. \tag{2.17}$$

Superprimaries obeying (2.10) are obtained by a similarity transformation as

$$|\psi_i\rangle = \begin{pmatrix} \psi_U \\ \psi_L \end{pmatrix} = \begin{pmatrix} \hat{\psi}_i e^{-\frac{1}{2}e^{2X^0}-(\frac{N-1}{2}+\lambda_i)X^0} \\ \hat{\psi}_i e^{-\frac{1}{2}e^{2X^0}-(\frac{N-1}{2}-\lambda_i)X^0} \end{pmatrix}. \tag{2.18}$$

Normalizability then requires convergence of the integral

$$\langle\psi_j|\psi_k\rangle = \int d^N X \sqrt{g}\psi_j^\dagger \psi_k = \delta_{jk} 2^{\frac{1-N}{2}} \int dX^0 (2\sinh 2\lambda_j X^0) e^{[X^0 - e^{2X^0}]}, \tag{2.19}$$

which is equivalent to the condition that $|\lambda_j| < 1/2$. We conclude that there is one superprimary for every such normalizable solution of the Dirac equation with torsion on the transverse space M^{N-1}.

3. SU(1, 1|1)

In this section we show that states of SU(1, 1|1) superconformal quantum mechanics are naturally viewed as $(p, 0)$-forms, but with a non-canonical measure. The chiral primary condition is derived and expressed as a condition on forms.

3.1 Symmetries

The SU(1, 1|1) algebra is

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3.1}$$

$$\{G_r, \bar{G}_s\} = 2L_{r+s} + 2J(r-s)\delta_{r,-s} \tag{3.2}$$

$$[J, G_r] = \frac{1}{2}G_r\,, \qquad [J, \bar{G}_r] = -\frac{1}{2}\bar{G}_r \tag{3.3}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}\,, \qquad [L_m, \bar{G}_r] = \frac{m-2r}{2}\bar{G}_{m+r} \tag{3.4}$$

where $m, n = 0, \pm 1$ and $r, s = \pm 1/2$. As shown in [11], the OSp(1|2) model of the previous section has this larger symmetry if and only if there is a complex structure preserved by the action of D, the metric is hermitean, and the $(1, 2)$ part of the torsion is given by

$$c^{\bar{a}}{}_{\bar{b}c} = -\Gamma^{\bar{a}}{}_{\bar{b}c}\,. \tag{3.5}$$

Indices in lower (upper) case denote complex (real) coordinates, so that $a, b = 1, 2, \ldots, n$, where n is the complex dimension. In general c may also have $(0, 3)$ and $(3, 0)$ parts unconstrained by (3.5). These are constrained to vanish for $\mathcal{N} = 4$

supersymmetry, and for simplicity we set them to zero here. The relation (3.5) then implies that there is a U(n) connection,

$$\Omega^N{}_{MP} \equiv \Gamma^N{}_{MP} + c^N{}_{MP} . \tag{3.6}$$

Let us first collect some formulae from [11] describing the SU(1,1|1) theories.[1] It is convenient to define the shifted momentum

$$\Pi_M = P_M - \frac{i}{2}(\omega_{MNP} - c_{MNP})\lambda^N\lambda^P , \tag{3.7}$$

which has the property

$$[\Pi_M, \lambda^N] = i\Omega^N{}_{MP}\lambda^P . \tag{3.8}$$

Defining Q, S and their hermitean conjugates $\bar{Q}$, $\bar{S}$ by

$$\begin{aligned} G_{\pm 1/2} &= \frac{1}{\sqrt{2}}(Q \mp iS) , \\ \bar{G}_{\pm 1/2} &= \frac{1}{\sqrt{2}}(\bar{Q} \mp i\bar{S}) , \end{aligned} \tag{3.9}$$

and

$$c_a = c^b{}_{ba} , \tag{3.10}$$

one has

$$\begin{aligned} Q &= \lambda^a(\Pi_a - ic_{ab\bar{c}}\lambda^b\lambda^{\bar{c}} - ic_a) , \\ S &= \lambda^a D_a . \end{aligned} \tag{3.11}$$

We note that

$$\begin{aligned} \{Q, \lambda^a\} &= 0 , \\ \{Q, \lambda^{\bar{a}}\} &= g^{\bar{a}b}(\Pi_b - ic_b) . \end{aligned} \tag{3.12}$$

3.2 The ground state of H

In this subsection we construct a state $|\eta\rangle$ annihilated by the supercharges Q and $\bar{Q}$ as well as the hamiltonian H. We compute the norm of $|\eta\rangle$ and its U(1) charge.

We begin by imposing the chirality conditions

$$\gamma^{\bar{a}}|\eta\rangle = \gamma_a|\eta\rangle = 0 , \tag{3.13}$$

[1] Our notation is the same as in [11] with the following exceptions. We use capital indices M, N, P for the $2n$-dimensional moduli space coordinates. The indices A, B are used only for the black holes themselves. In this section, Q and S are the holomorphic supercharges that were called $\mathcal{Q} = \frac{1}{2}(Q - i\tilde{Q})$ and $\mathcal{S} = \frac{1}{2}(S - i\tilde{S})$ in [11]. In section 4, R and J_R are the angular momentum operators corresponding to R_+ and R_- in [11].

JHEP03(2001)050

which states that $|\eta\rangle$ is a singlet under the SU(n) subgroup of U(n). It also implies that $|\eta\rangle$ is annihilated by $\bar{Q}$ (as well as $\bar{S}$). The action of the supercharge Q on a general state $|\psi\rangle$ is

$$Q|\psi\rangle = -i\gamma^a(\mathcal{D}_a + c_{ab\bar{c}}\gamma^b\gamma^{\bar{c}} + c_a)|\psi\rangle\,, \tag{3.14}$$

where $\mathcal{D}$ is the covariant derivative on spinors with the U(n) spin connection associated to Ω. Given (3.13), the state $|\eta\rangle$ will be annihilated by Q if the wavefunction $\eta(X)$ is a covariantly constant spinor obeying

$$(\mathcal{D}_a + c_a)\eta = 0\,. \tag{3.15}$$

This has solutions because the integrability conditions $[\mathcal{D}_a+c_a, \mathcal{D}_b+c_b]\eta = -2c^c{}_{ab}(\mathcal{D}_c+c_c)\eta$ are satisfied, as can be checked from the expression (3.6) for the connection. The solution is fixed up to transformations of the form

$$\eta \to f(X^{\bar{a}})\eta\,, \tag{3.16}$$

where f is an antiholomorphic function. We will fix this freedom shortly.

We now describe some properties of $|\eta\rangle$. Define the U(1) connection A by

$$A_N = \Omega^M{}_{NP}I_M{}^P = i\Omega^a{}_{Na} - i\Omega^{\bar{a}}{}_{N\bar{a}}\,, \tag{3.17}$$

where I is the complex structure. A obeys

$$A_b = -4ic_b + 2i\partial_b\phi\,, \tag{3.18}$$

with

$$\phi = \frac{1}{4}\ln\det g\,. \tag{3.19}$$

Under an infinitesimal complex coordinate transformation

$$X^a \to X^a + \zeta^a(X^b)\,, \tag{3.20}$$

one finds

$$\begin{aligned} \phi &\to \phi + \frac{i}{2}(\epsilon - \bar{\epsilon}) \\ A &\to A - d(\epsilon + \bar{\epsilon})\,, \end{aligned} \tag{3.21}$$

with complex holomorphic gauge parameter

$$\epsilon = i\partial_a\zeta^a\,. \tag{3.22}$$

Using the fact that the spinorial wavefunction η is an SU(n) (but not a U(1)) singlet, the equation (3.15) can be written, using (3.18), as

$$\left(\partial_a - \frac{i}{4}A_a + c_a\right)\eta = \left(\partial_a + \frac{1}{2}\partial_a\phi\right)\eta = 0\,. \tag{3.23}$$

JHEP03(2001)050

The solution of this equation is

$$\eta = e^{-\phi/2+\bar{\Theta}}\eta_0\,, \tag{3.24}$$

where $\bar{\Theta}$ is an arbitrary antiholomorphic function, and η_0 is the constant spinor obeying (3.13). The norm of η is then

$$\eta^\dagger\eta = e^{-\phi+\Theta+\bar{\Theta}}\,. \tag{3.25}$$

The expression (3.25) is coordinate invariant because $\Theta+\bar{\Theta}$ and ϕ transform the same way under (3.20). In the following it will be convenient to choose coordinates so that ϕ is nonsingular at smooth points in the geometry. (In the $\mathcal{N}=4$ case, such coordinates are singled out by the existence of a quaternionic structure.) The norm (3.25) is then in general singular, except if the freedom (3.16) is used to shift away Θ altogether. We shall henceforth assume that this has been done, so that in nonsingular coordinates

$$\eta^\dagger\eta = e^{-\phi}\,. \tag{3.26}$$

The U(1) R-charge is measured by the operator

$$J = \frac{1}{2}\left(iD^a\Pi_a - iD^{\bar{a}}\Pi_{\bar{a}} + g_{a\bar{b}}(\lambda^a\lambda^{\bar{b}} - \lambda^{\bar{b}}\lambda^a) + D^a c_{aMN}\lambda^M\lambda^N - D^{\bar{a}}c_{\bar{a}MN}\lambda^M\lambda^N\right). \tag{3.27}$$

Acting on $|\eta\rangle$, one finds

$$J|\eta\rangle = \frac{1}{2}(D^{\bar{a}}\partial_{\bar{a}}\phi - n)|\eta\rangle\,. \tag{3.28}$$

In dilational gauge, defined by

$$D^M = \frac{2}{h}X^M\,, \tag{3.29}$$

for some constant h, one has

$$D^{\bar{a}}\partial_{\bar{a}}\phi = \frac{n(h-2)}{2h}\,, \tag{3.30}$$

and

$$J|\eta\rangle = -n\frac{h+2}{4h}|\eta\rangle\,, \tag{3.31}$$

where n is the complex dimension of the target space.

3.3 Chiral primaries and $(p,0)$-forms

In the previous subsection we constructed a supersymmetric ground state $|\eta\rangle$ of the hamiltonian H. In general this state may not be normalizable due to the noncompact regions of the target space. L_0 eigenstates will in some cases be normalizable. In this subsection we build such states by acting on $|\eta\rangle$ with bosonic and fermionic operators.

JHEP03(2001)050

As in the case of OSp(1|2), L_0 eigenstates lie in superconformal representations containing lowest weight states annihilated by $G_{1/2}$ and $\bar{G}_{1/2}$. For SU(1,1|1) there are special representations whose lowest weight states are also annihilated by $G_{-1/2}$ (or $\bar{G}_{-1/2}$). These are chiral (or antichiral) primary states.

Consider a similarity transformation from $|\eta\rangle$ to the state

$$|0\rangle = e^{-K}|\eta\rangle\,. \tag{3.32}$$

Using

$$G_{1/2} = \frac{1}{\sqrt{2}}e^{-K}Qe^{K}\,, \tag{3.33}$$

and (3.13), this state is seen to obey

$$G_{1/2}|0\rangle = \bar{G}_{\pm 1/2}|0\rangle = 0\,. \tag{3.34}$$

Hence it is a lowest weight antichiral primary. We shall see later that this state is not normalizable in the black hole case.

The most general state is of the form

$$|f_p\rangle = f_p|0\rangle\,, \tag{3.35}$$

with

$$f_p = \frac{1}{p!}f_{a_1a_2\ldots a_p}\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_p}\,, \tag{3.36}$$

where $f_{a_1a_2\ldots a_p}$ is totally antisymmetric. Hence the Hilbert space can be identified with the space of $(p,0)$-forms on the target space. The action of the supercharges on these states is[2]

$$(Q-iS)f_p|0\rangle = -\frac{i}{p!}(\partial_{a_1}f_{a_2\ldots a_{p+1}})\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_{p+1}}|0\rangle\,, \tag{3.37}$$

$$(\bar{Q}-i\bar{S})f_p|0\rangle = -\frac{i}{(p-1)!}(e^{\phi}\nabla^{M}e^{-\phi}f_{Ma_2\ldots a_p})\lambda^{a_2}\cdots\lambda^{a_p}|0\rangle\,, \tag{3.38}$$

$$Sf_p|0\rangle = \frac{1}{p!}D_{a_1}f_{a_2\ldots a_{p+1}}\lambda^{a_1}\cdots\lambda^{a_{p+1}}|0\rangle\,, \tag{3.39}$$

$$\bar{S}f_p|0\rangle = \frac{1}{(p-1)!}D^{a_1}f_{a_1a_2\ldots a_p}\lambda^{a_2}\cdots\lambda^{a_p}|0\rangle\,. \tag{3.40}$$

Regarding f_p as a $(p,0)$-form, such states are chiral primary if and only if

$$\partial f_p = D\wedge f_p = \bar{\partial} * e^{-\phi}f_p = 0\,, \tag{3.41}$$

where $*$ is the Hodge dual and here $D = D_a dX^a$ is a $(1,0)$-form.

Note that the inner product,

$$\langle f'_p|f_p\rangle = \frac{1}{p!}\int d^{2n}x\sqrt{g}e^{-\phi-2K}\bar{f}'^{a_1a_2\ldots a_p}f_{a_1a_2\ldots a_p}\,, \tag{3.42}$$

contains extra factors in the integration measure.

[2]The use of a capital index in the second equation produces one extra term. For example, $\nabla^M f_M = g^{a\bar{b}}(\partial_{\bar{b}}f_a - 2\Gamma^c{}_{\bar{b}a}f_c)$, while $\nabla^a f_a = g^{a\bar{b}}(\partial_{\bar{b}}f_a - \Gamma^c{}_{\bar{b}a}f_c)$.

3.4 $\mathrm{Tr}(-)^{2J}$

In this subsection we consider the Witten index

$$\chi = \mathrm{Tr}(-)^{2J} \,. \tag{3.43}$$

In order to compute the trace one must choose a basis for the Hilbert space and a regulator for the infinite sum. One might attempt to define the trace as the limit of a weighted sum over H eigenstates

$$\chi = \mathrm{Tr}_H (-)^{2J} e^{-\beta H} \,, \qquad \beta \to 0 \,. \tag{3.44}$$

Of course the sum is actually independent of β, since states with nonzero H come in bose-fermi pairs which cancel. The trace (3.44) is nevertheless difficult to evaluate because of the continuum of eigenstates extending down to zero energy. An alternate way to define the index is as a weighted sum over L_0 eigenstates[3]

$$\chi = \mathrm{Tr}_{L_0} (-)^{2J} e^{-\beta (L_0 - J)} \,, \qquad \beta \to 0 \,. \tag{3.45}$$

The sum is also independent of β. This follows from considering the operator

$$\tilde{Q} = \frac{1}{\sqrt{2}} (\bar{G}_{1/2} + G_{-1/2}) \tag{3.46}$$

with the properties

$$\tilde{Q}^2 = L_0 - J \,, \qquad \{\tilde{Q}, (-)^{2J}\} = 0 \,. \tag{3.47}$$

States with nonzero $L_0 - J$ come in bose-fermi pairs generated by $\tilde{Q}$ and cancel in the sum (3.45). Hence χ receives contributions only from states with $L_0 - J = 0$. Such states are chiral primaries annihilated by $\tilde{Q}$, or equivalently, states annihilated by $\bar{G}_{1/2}$ and $G_{\pm 1/2}$. Hence the Witten index can be computed as a weighted sum over superconformal chiral primaries.

A more general index can be defined when the theory contains an operator $\mathcal{O}$, usually associated with a symmetry, which commutes with the generators of the superalgebra. In that case the preceding argment may be repeated to show that

$$\chi_{\mathcal{O}} = \mathrm{Tr}_{L_0} (-)^{2J} \mathcal{O} e^{-\beta (L_0 - J)} \,, \qquad \beta \to 0 \tag{3.48}$$

also defines an index. Alternatively the index can be restricted to a sum over the eigenstates of $\mathcal{O}$ with eigenvalue λ

$$\chi(\lambda) = \mathrm{Tr}_{L_0} (-)^{2J} e^{-\beta (L_0 - J)} |_{\lambda} \,, \qquad \beta \to 0 \tag{3.49}$$

[3] In $1+1$ dimensions, (3.45) is exactly the expression obtained for the Witten index in the NS sector obtained by spectral flow from the R sector. In 0+1 there is no obvious analog of spectral flow.

4. $D(2,1;0)$

In this section we describe $D(2,1;0)$ quantum mechanics in terms of its symmetries and chiral primaries. This algebra contains an $\mathrm{SU}(1,1|1)$ subalgebra and will be used in section 5 to describe the theory of N BPS black holes.

4.1 Symmetries

$D(2,1;0)$ is the semidirect product of $\mathrm{SU}(1,1|2)$ and $\mathrm{SU}(2)_R$. In $4k$ dimensions the target space geometry has a triplet of self-dual complex structures obeying

$$I^r I^s = -\delta^{rs} + \epsilon^{rst} I^t \,, \tag{4.1}$$

for $r, s = 1, 2, 3$. There are also isometries generated by

$$D^{rM} = D^N I_N^{rM} \,, \tag{4.2}$$

whose associated charges generate the R-symmetry in $\mathrm{SU}(1,1|2)$. This R-symmetry will be denoted by $\mathrm{SU}(2)_{\mathrm{Right}}$. A large class of $D(2,1;0)$ theories are sigma models with target space metrics

$$g_{a\bar{b}} = \frac{1}{2}(\partial_a \partial_{\bar{b}} L + I_a^{-\bar{c}} I_{\bar{b}}^{+d} \partial_{\bar{c}} \partial_d L) \,. \tag{4.3}$$

In this expression, L is a homogeneous function of degree -2, the complex coordinates are adapted to I^3, and $I^\pm = \frac{1}{2}(I^1 \pm i I^2)$. All three complex structures are constant self-dual matrices, and $D^M = -X^M$.

It follows from (4.3) that

$$\partial_a \phi = 2 c_a \,. \tag{4.4}$$

Hence for $D(2,1;0)$ the $\mathrm{U}(1)$ connection A in (3.17) vanishes, and Ω is an $\mathrm{SU}(n)$ (rather than $\mathrm{U}(n)$) connection. The holomorphic two-form I^- obeys the relations

$$\partial I^- = 0 \,, \qquad \bar{\partial} * e^{-\phi} I^- = 0 \,. \tag{4.5}$$

There are 8 supercharges in $\mathrm{SU}(1,1|2)$,

$$G^{\pm\pm}_{\pm 1/2} = \frac{1}{\sqrt{2}}(Q^{\pm\pm} \mp i S^{\pm\pm}) \,. \tag{4.6}$$

Explicit expressions for four of these are

$$\begin{aligned} G^{+-}_{\pm 1/2} &= \lambda^a \left(\Pi_a - i c_{ab\bar{c}} \lambda^b \lambda^{\bar{c}} - i c_a \mp i D_a\right) , \\ G^{-+}_{\pm 1/2} &= \lambda^{\bar{a}} \left(\Pi_{\bar{a}} - i c_{\bar{a}b\bar{c}} \lambda^b \lambda^{\bar{c}} - i c_{\bar{a}} \mp i D_{\bar{a}}\right) . \end{aligned} \tag{4.7}$$

These charges are $SU(2)_{\rm Right} \times SU(2)_R$ doublets. The $SU(2)_{\rm Right}$ ($SU(2)_R$) spin is indicated by the first (second) superscript. The other four supercharges can accordingly be expressed as

$$G^{++}_{\pm1/2} = i[J^+_R, G^{-+}_{\pm1/2}] = -i[R^+, G^{+-}_{\pm1/2}]\,, \tag{4.8}$$
$$G^{--}_{\pm1/2} = i[J^-_R, G^{+-}_{\pm1/2}] = -i[R^-, G^{-+}_{\pm1/2}]\,. \tag{4.9}$$

Some important commutators are

$$\begin{aligned}
\{G^{\alpha\alpha'}_p, G^{\beta\beta'}_q\} &= 2\delta^{\alpha,-\beta}\delta^{\alpha',-\beta'}L_{p+q} + 2(p-q)J^{\alpha\beta}_R\delta^{\alpha',-\beta'}\,,\\
[J^3_R, G^{\alpha\alpha'}_p] &= \frac{\alpha}{2}G^{\alpha\alpha'}_p\,,\\
[R^3, G^{\alpha\alpha'}_p] &= \frac{\alpha'}{2}G^{\alpha\alpha'}_p\,,
\end{aligned} \tag{4.10}$$

with $p,q = \pm1/2$, $\alpha\beta = \pm$ and $J^{+-}_R = -J^{-+}_R = J^3_R$ etc. J_R generates $SU(2)_{\rm Right}$ rotations of the bosons X^A,

$$[J^r_R, \lambda^M] = 0\,, \qquad [J^r_R, X^M] = \frac{i}{2}X^N I^{rM}_N\,. \tag{4.11}$$

The second $SU(2)_R$, which does not lie in $SU(1,1|2)$, generates R-symmetry transformations of the fermions.[4]

$$[R^r, \lambda^M] = \frac{i}{2}\lambda^N I^{rM}_N\,, \qquad [R^r, X^M] = 0\,. \tag{4.12}$$

The supercharges transform in the $(2,2)$ of $SU(2)_{\rm Right} \times SU(2)_R$.

4.2 Chiral primaries

Chiral primaries in these theories are defined by embedding an $SU(1,1|1)$ subalgebra in $D(2,1;0)$. A natural embedding is defined by

$$G_{\pm1/2} = G^{+-}_{\pm1/2}\,, \qquad \bar{G}_{\pm1/2} = G^{-+}_{\pm1/2}\,, \qquad J = J^3_R\,. \tag{4.13}$$

With this embedding, the conditions for a chiral primary are given by (3.41), in the complex structure defined by I^3.

It follows from the algebra that normalizable chiral primaries annihilated by $\bar{G}_{1/2}$ and $G_{\pm1/2}$ as in (3.34) are annihilated by six of the eight supercharges

$$G^{++}_{\pm1/2}\,, \qquad G^{+-}_{\pm1/2}\,, \qquad G^{-+}_{1/2}\,, \qquad G^{--}_{1/2}. \tag{4.14}$$

Antichiral primaries are annihilated by the complementary set of negatively moded supercharges. Note that $SU(2)_R$ does not mix the operators in (4.14) with $G^{-\pm}_{-1/2}$. Therefore if a chiral primary is not an $SU(2)_R$ singlet, then the $SU(2)_R$ action gives new chiral primaries. The algebra further implies that all chiral primaries are highest weight $SU(2)_{\rm Right}$ states annihilated by J^+_R.

[4] (R, J_R) are the operators (R_+, R_-) of [11].

JHEP03(2001)050

We note that the state $|0\rangle$ is annihilated by all the positively moded supercharges as well as $G^{\pm+}_{-1/2}$. Hence it is neither a chiral nor an antichiral primary. This implies it cannot be a normalizable state, as indeed can be seen from explicit computation. Since $h = -2$ for $D(2,1;0)$, it follows from (3.31) that $J^3_R|0\rangle = 0$, while $R^3|0\rangle = \frac{n}{4}|0\rangle$.

5. Black hole quantum mechanics

At low energies the quantum mechanics of N black holes is described by the product of a free theory containing the center-of-mass coordinates and an interacting near-horizon superconformal theory with a $4(N-1)$-dimensional target space. In this section we describe this theory and the index $\mathcal{I}_{BH}$ that counts the weighted number of supersymmetric bound states. For the case of $N = 2$ we find all the $j_L = 0$ chiral primaries and discover $\mathcal{I}_{BH}(0) = -1$.

5.1 The center-of-mass multiplet

Throughout this paper we have largely neglected the center-of-mass degrees of freedom. We give a brief description here for completeness. The center-of-mass theory contains two complex bosons X^k in the (2,2) of $\mathrm{SU}(2)_{\mathrm{Left}} \times \mathrm{SU}(2)_{\mathrm{Right}}$ and two complex fermions λ^k in the (2,2) of $\mathrm{SU}(2)_{\mathrm{Left}} \times \mathrm{SU}(2)_R$. The ground state is therefore a spinor of $\mathrm{SU}(2)_{\mathrm{Left}} \times \mathrm{SU}(2)_R$. Explicitly, defining $|\eta\rangle$ as a zero momentum state obeying

$$\lambda^{\bar{k}}|\eta\rangle = 0\,, \tag{5.1}$$

there is an $\mathrm{SU}(2)_R$ doublet of spacetime bosons

$$|\eta\rangle\,, \qquad \lambda^1\lambda^2|\eta\rangle\,, \tag{5.2}$$

and an $\mathrm{SU}(2)_{\mathrm{Left}}$ doublet of spacetime fermions

$$\lambda^1|\eta\rangle\,, \qquad \lambda^2|\eta\rangle\,. \tag{5.3}$$

This is exactly the content of a massive, positively charged, spacetime hypermultiplet. These states are all annihilated by the four supercharges Q^α because $P_k = 0$.

5.2 The superconformal sector

The low energy interactions between N five-dimensional BPS black holes with charges Q_A are described by the $4(N-1)$-dimensional $D(2,1;0)$ theory with metric (4.3) constructed from the potential [1]

$$L = -\int d^4X \left(\sum_{A=1}^{N} \frac{Q_A}{|\vec{X} - \vec{X}^A|^2}\right)^3. \tag{5.4}$$

In this context $\mathrm{SU}(2)_{\mathrm{Right}}$ in $\mathrm{SU}(1,1|2)$ generates right-handed spatial rotations [12], while $\mathrm{SU}(2)_R$ generates the spacetime R-symmetry transformations.

The theory following from (5.4) has additional global symmetries corresponding to $\mathrm{SU}(2)_{\mathrm{Left}}$ spatial rotations. These are generated by

$$J_L^r = -\frac{1}{2} X^M K^r_M{}^N \left(\Pi_N - \frac{i}{2} \Omega^P{}_{NQ} \lambda^Q \lambda_P \right) + \frac{i}{4} K^r_{MN} \lambda^M \lambda^N \,, \tag{5.5}$$

where K^r are the constant anti-self dual complex structures on $\mathbb{R}^4$. J_L obeys

$$[J_L^r, \lambda^M] = \frac{i}{2} \lambda^N K^{rM}_N \,, \qquad [J_L^r, X^M] = \frac{i}{2} X^N K^{rM}_N \,, \qquad [J_L^r, J_R^s] = 0 \,, \tag{5.6}$$

and commutes with all the generators of the superconformal group. λ^M transforms in the $(1,2,2)$ of $\mathrm{SU}(2)_{\mathrm{Right}} \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_{\mathrm{Left}}$, as appropriate for a goldstino. The full symmetry group of the system is $\mathrm{SU}(2)_{\mathrm{Left}} \times D(2,1;0)$.

5.3 Validity of the approximations

In this subsection we discuss potential corrections to the theory defined by (5.4), and in particular whether or not they could affect the conclusion that there is a divergent continuum of infrared states.

The superconformal theory defined by (5.4) was derived in [1] from a more general (non-superconformal) quantum mechanics, in which a constant is added to the sum inside the parentheses. Eq. (5.4) then arises in an $M_p \to \infty$ limit [7, 1], with the rescaled separations $|\vec{X}^A - \vec{X}^B| \to M_p^{3/2} |\vec{X}^A - \vec{X}^B|$ (with dimensions $\sqrt{\text{mass}}$) held fixed. At the same time the energies are rescaled by a factor of M_p so that, in the limit, all excitations of the superconformal theory have zero energy as measured with respect to the original time coordinate at spatial infinity. We wish to know whether all of these zero-energy states are really present or whether some of them might be removed by corrections which have been neglected so far.

Since M_p has been taken to infinity, there can be no $1/M_p$ corrections to (5.4). As there are no dimensionful parameters in the infrared limit, corrections to (5.4) must be supressed by dimensionless quantities such as $\dot{X}/X^3$. Such terms can indeed be seen to arise for example as Born-Infeld type corrections. For the case of two black holes the corrected action is of the general form

$$\frac{1}{2} \int dt \left(\frac{|\partial_t \vec{X}|^2}{|\vec{X}|^4} + \frac{|\partial_t \vec{X}|^4}{|\vec{X}|^{10}} \cdots \right) , \tag{5.7}$$

where $\vec{X}$ is the relative separation. In terms of the momentum defined from the leading term

$$\vec{P} = \frac{\partial_t \vec{X}}{|\vec{X}|^4} \,, \tag{5.8}$$

this becomes

$$\frac{1}{2} \int dt \left(|\vec{X}|^4 |\vec{P}|^2 + |\vec{X}|^6 |\vec{P}|^4 + \cdots \right) . \tag{5.9}$$

Such correction terms can be neglected as long as

$$|\vec{P}| \ll \frac{1}{|\vec{X}|}\,. \tag{5.10}$$

As the black holes approach one another, the relative momenta must be smaller and smaller in order to suppress corrections. Hence we do not expect all states to be reliably described by the superconformal theory.

The number of states which can be reliably described by the superconformal theory can be estimated by the volume Ω of phase space in which (5.10) is obeyed. This is, with an infrared cutoff $\epsilon \to 0$,

$$\Omega \sim \int_{|\vec{X}|>\epsilon} d^4X \int_{|\vec{P}|<1/|\vec{X}|} d^4P \sim \ln\epsilon\,. \tag{5.11}$$

Hence, according to this rather crude estimate, a logarithmic infrared divergence in the number of states appears to remain even when the untrustworthy regions of phase space are removed.

We can also compute the density of states as function of the energy. This is

$$d\Omega \sim \int d^4X d^4P \delta(E - |\vec{X}|^4|\vec{P}|^2)dE\,. \tag{5.12}$$

This is divergent for any E if one does not impose the restriction (5.10). Imposing (5.10) leads to the finite, scale invariant, result,

$$d\Omega \sim \frac{dE}{E}\,. \tag{5.13}$$

Since this density of (reliably present) states is finite, it is possible that the infrared divergences do not appear in physical processes involving scattering off of the collection of black holes. A similar mechanism was discussed in [13].

Another potential source of corrections comes from black hole fragmentation as in [7]. The moduli space geometry (4.3), and (5.4) was derived using the low-energy supergravity approximation. The validity of this requires that the spacetime curvature is small compared to the inverse Planck length, or equivalently $Q_A \gg 1$.[5] In this paper we have ignored the possibility that the black holes might fragment into smaller pieces. In five dimensions we know of no way to suppress this energetically. One might try to avoid this by taking all the black holes to carry the minimum quantum of charge. However it is not clear whether the expression for the moduli space metric remains valid for small charges. The expression is highly constrained both by the symmetries and the known long-distance behavior. Whether or not corrections do appear at small charge is an open question which we shall not attempt to resolve here.

[5]Perhaps surprisingly, if $Q_A \gg 1$, then the curvatures remain small even for $\vec{X}^A - \vec{X}^B \to 0$.

In four dimensions the situation is better. Let all the black holes carry the same charges, with large, nonzero coprime electric and magnetic charges. In that case the possiblity of fragmentation is eliminated energetically, as can be seen from the BPS mass formula. The moduli space geometry [10, 4] and the bound-state analysis is similar for this case and will appear elsewhere.

5.4 $\mathcal{I}_{BH}$

In this section we relate a spacetime index counting weighted degeneracies of spacetime BPS multiplets to an index in the superconformal quantum mechanics of the type discussed in subsection 3.4.

Massive representations of $\mathcal{N}=2$ Poincaré supersymmetry in five dimensions are of two types. The generic representation is the long multiplet L_j, with $\mathrm{SU}(2)_{\mathrm{Left}} \times \mathrm{SU}(2)_{\mathrm{Right}}$ spin content

$$L_j: \quad [j_L, j_R] \otimes ([1/2,1/2] + 2[1/2,0] + 2[0,1/2] + 4[0,0]). \tag{5.14}$$

This multiplet has $8(2j_L+1)(2j_R+1)$ bosons, $8(2j_L+1)(2j_R+1)$ fermions, and maximal spin $(j_L+1/2, j_R+1/2)$. There is also a short multiplet S_j which is annihilated by half of the supercharges and has Mass = Charge in appropriate units. This multiplet has spin content

$$S_j: \; [j_L, j_R] \otimes ([1/2,0] + 2[0,0])\,. \tag{5.15}$$

This multiplet has one quarter[6] as many bosons and fermions, and maximal spin $(j_L+1/2, j_R)$. It is easy to see that for either multiplet

$$\mathrm{Tr}(-)^{2J_L^3+2J_R^3} = 0\,. \tag{5.16}$$

An alternative index,

$$\mathcal{I} \equiv \mathrm{Tr}(-)^{2J_R^3} y^{2J_L^3}\,, \tag{5.17}$$

vanishes for long multiplets

$$\mathcal{I}(L_j) = 0\,, \tag{5.18}$$

but not for short ones:

$$\mathcal{I}(S_j) = (-)^{2j_R}(2j_R+1)\frac{(y^{1/2}+y^{-1/2})^2}{y-y^{-1}}(y^{2j_L+1} - y^{-2j_L-1})\,. \tag{5.19}$$

The value of this index traced over all the quantum states of N black holes gives a measure of the weighted number of supersymmetric states.

[6]We do not include the conjugate multiplet with negative charges.

In the supersymmetric quantum mechanics, j_L and j_R are the eigenvalues of the operators J_L^3 (equation (5.5)) and J_R^3 (equation (3.27) specialized to the black hole case),[7] augmented by the corresponding operators for the center-of-mass multiplet. The trace over this latter multiplet gives a universal factor of $(y^{1/2}+y^{-1/2})^2$. The total index $\mathcal{I}_{BH}$ counting the weighted number of supersymmetric black hole bound states is then defined by

$$\mathcal{I}_{BH}^{\rm tot} = (y^{1/2}+y^{-1/2})^2 \, {\rm Tr}_{SCQM}(-)^{2J_R^3} y^{2J_L^3} \,, \tag{5.20}$$

where the trace is over the internal Hilbert space of the superconformal quantum mechanics, without the center-of-mass multiplet. We may also define a reduced index of the form (3.45) by factoring out the center of mass factor and restricting to the subspace transforming in the dimension $2j_L+1$ representation of $\mathrm{SU}(2)_L$, as

$$\mathcal{I}_{BH}(j_L) = {\rm Tr}_{SCQM}(-)^{2J_R^3}|_{j_L} \,. \tag{5.21}$$

In the next section we will evaluate this for two black holes and $j_L = 0$ by counting chiral primaries.

5.5 $N=2$

For the case of two black holes, the $D(2,1;0)$ quantum mechanics governing their relative separation $\vec{X}_{12} \equiv \vec{X}^1 - \vec{X}^2$ is described by the metric

$$ds^2 = 12\pi^2 (Q_1^2 Q_2 + Q_1 Q_2^2) \frac{d\vec{X}_{12}\cdot d\vec{X}_{12}}{|\vec{X}_{12}|^4} \,, \tag{5.22}$$

as follows from (4.3) and (5.4). One finds

$$\vec{D} = -\vec{X}_{12} \,, \qquad K = \frac{6\pi^2(Q_1^2 Q_2 + Q_2^2 Q_1)}{|\vec{X}_{12}|^2} \,, \qquad e^{-\phi} = \frac{|\vec{X}_{12}|^4}{6\pi^2(Q_1^2 Q_2 + Q_2^2 Q_1)} \,. \tag{5.23}$$

There are states constructed as in (3.36) corresponding to $(p,0)$-forms with $p = 0,1,2$. For $p=0$ the chiral primary condition (3.41) reduces to $\partial f_0 = 0$ and $D f_0 = 0$, which has no nontrivial solutions.

For $p=1$ the condition $D \wedge f_1 = 0$ implies that f_1 is proportional to D. The condition $\partial f_1 = 0$ then implies that the proportionality factor is a function of K times an antiholomorphic function. In complex coordinates (z^1, z^2)

$$f_1 = c(\bar{z}^1, \bar{z}^2, K) D \,. \tag{5.24}$$

For the case considered here of $j_L = 0$, f_1, and therefore c, must be invariant under $\mathrm{SU}(2)_L$ rotations. This requires c to be a function of K only. Using the formula (3.30) for the special case $n=2$ and $h=-2$, one finds

$$\bar{\partial} * e^{-\phi} D = 0 \,. \tag{5.25}$$

[7] We recall that J_R is part of the superconformal algebra, while J_L commutes with it.

It follows that the last condition in (3.41) is satisfied only when the proportionality factor is a constant. Hence the state

$$|D\rangle = D_a \lambda^a |0\rangle \tag{5.26}$$

is the unique chiral primary with $p = 1$ for $N = 2$. The norm is

$$\langle D|D\rangle = 2\int d^4X\sqrt{g}e^{-\phi}Ke^{-2K}\,. \tag{5.27}$$

For a pair of black holes K goes like $1/r^2$, and so (5.27) converges at both large and small r.[8] The state $|D\rangle$ is an $\mathrm{SU}(2)_R$ singlet and the $J_R^3 = +1/2$ element of an $\mathrm{SU}(2)_{\mathrm{Right}}$ doublet.

At $p = 2$, the first two chiral primary conditions are trivially satisfied. The general solution of the last condition (using the relations (4.5)) is the $(2,0)$-form

$$f(X)I^-_{ab}\,, \tag{5.28}$$

where $f(X)$ is holomorphic. Eq. (5.28) is singular unless f is constant. In this case the norm becomes

$$\langle I^-|I^-\rangle = 2\int d^4X\sqrt{g}e^{-\phi-2K}\,, \tag{5.29}$$

where we have used $I^{+ab}I^-_{ab} = 2$. The integral in (5.29) diverges logarithmically at $x \to \infty$. Hence there are no normalizable chiral primaries at $p = 2$.

We conclude that the supersymmetric index $\mathcal{I}_{BH}(j_L = 0)$ is -1 for a pair of black holes.

Acknowledgments

We are grateful to J. Maldacena, A. Maloney, J. Michelson and M. Spradlin for useful conversations, and to M. Stern for pointing out an error in an earlier version of this work. This work is supported in part by DOE grant DE-FG02-91ER40654 and an NDSEG graduate fellowship. A.V. is also supported by INTAS-OPEN-97-1312.

References

[1] J. Michelson and A. Strominger, *Superconformal multi-black hole quantum mechanics*, *J. High Energy Phys.* **09** (1999) 005 [hep-th/9908044].

[2] R. Britto-Pacumio, J. Michelson, A. Strominger and A. Volovich, *Lectures on superconformal quantum mechanics and multi-black hole moduli spaces*, hep-th/9911066.

[8] It is now seen explicitly that the norm $\langle 0|0\rangle$ of the L_0 ground state diverges logarithmically at large r. This norm is given by the integral in (5.27) without the factor of K.

[3] J. Gutowski and G. Papadopoulos, *The dynamics of very special black holes*, *Phys. Lett.* **B 472** (2000) 45 [hep-th/9910022].

[4] J. Gutowski and G. Papadopoulos, *Moduli spaces for four- and five-dimensional black holes*, *Phys. Rev.* **D 62** (2000) 064023 [hep-th/0002242].

[5] G.W. Gibbons and P.K. Townsend, *Black holes and Calogero models*, *Phys. Lett.* **B 454** (1999) 187 [hep-th/9812034].

[6] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend and A. Van Proeyen, *Black holes and superconformal mechanics*, *Phys. Rev. Lett.* **81** (1998) 4553 [hep-th/9804177].

[7] J. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter fragmentation*, *J. High Energy Phys.* **02** (1999) 011 [hep-th/9812073].

[8] S. Cacciatori, D. Klemm and D. Zanon, *W_∞ algebras, conformal mechanics and black holes*, *Class. and Quant. Grav.* **17** (2000) 1731 [hep-th/9910065].

[9] G. Papadopoulos, *Conformal and superconformal mechanics*, *Class. and Quant. Grav.* **17** (2000) 3715 [hep-th/0002007].

[10] A. Maloney, M. Spradlin and A. Strominger, *Superconformal multi-black hole moduli spaces in four dimensions*, hep-th/9911001.

[11] J. Michelson and A. Strominger, *The geometry of (super)conformal quantum mechanics*, *Comm. Math. Phys.* **213** (2000) 1 [hep-th/9907191].

[12] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Black holes of $D = 5$ supergravity*, *Class. and Quant. Grav.* **16** (1999) 1 [hep-th/9810204].

[13] T. Banks, M. O'Loughlin and A. Strominger, *Black hole remnants and the information puzzle*, *Phys. Rev.* **D 47** (1993) 4476 [hep-th/9211030].